In this chapter and the next we focus on the fundamentals of consumption and capital accumulation in dynamic nonmonetary equilibrium models. We introduce basic models—in this chapter, the Ramsey infinite horizon optimizing model, and in the next, overlapping generations models with finite horizon maximizers—and begin to analyze economic issues such as how much interest rates affect savings and whether the choice between tax and deficit financing affects capital accumulation.

Individuals are assumed in this chapter to have an infinite horizon, or to live forever. The infinite horizon assumption turns out to have strong implications: together with the assumptions of competitive markets, constant returns to scale in production, and homogeneous agents, it typically implies that the allocation of resources achieved by a decentralized economy will be the same as that chosen by a central planner who maximizes the utility of the representative economic agent in the model. We demonstrate here the equivalence between the allocation of resources in the decentralized economy and in a planned economy.

We start this chapter by developing the Ramsey (1928) analysis of optimal economic growth under certainty, by deriving the intertemporal conditions that are satisfied on the optimal path that would be chosen by a central planner. We then show, in section 2.2, the equivalence of the optimal path to the equilibrium path of the decentralized economy. In section 2.3 we examine the effects of both lump-sum and capital taxation on the rate of saving and the equilibrium interest rate in the framework of the decentralized infinite horizon economy.

In section 2.4 we analyze the economy of a small country, showing how the evolution of the current account is determined by investment and saving behavior. We examine the response of the economy to supply shocks, showing under what circumstances a country will respond to an adverse shock by borrowing abroad.
In the final section, section 2.5, we discuss some of the special features and implications of the intertemporally separable utility function with constant rate of time preference used in the chapter, and examine alternative formulations.

2.1 The Ramsey Problem

Frank Ramsey\(^2\) posed the question of how much a nation should save and solved it using a model that is now the prototype for studying the optimal intertemporal allocation of resources. The model presented in this section is essentially that of Ramsey.

The population, \(N_t\), grows at rate \(n\); it can be thought of as a family, or many identical families, growing over time. The labor force is equal to the population, with labor supplied inelastically. Output is produced using capital, \(K\), and labor. There is no productivity growth.

The output is either consumed or invested, that is, added to the capital stock. Formally,

\[
Y_t = F(K_t, N_t) = C_t + \frac{dK_t}{dt}.
\]

For simplicity, we assume that there is no physical depreciation of capital, or that \(Y_t\) is net rather than gross output. The production function is homogeneous of degree one: that is, there are constant returns to scale.

In per capita terms

\[
f(k_t) = c_t + \frac{dk_t}{dt} + nk_t.
\]

where lowercase letters denote per capita (equal to per worker) values of variables so that \(k\) is the capital-labor ratio and \(f(k_t) \equiv F(K_t/N_t, 1)\); we assume \(f(\cdot)\) to be strictly concave and to satisfy the following conditions, known as Inada conditions:

\[
f(0) = 0, \quad f'(0) = \infty, \quad f'(\infty) = 0.
\]

We also assume that the economy starts with some capital so that it can get production off the ground:

\[
k_0 > 0.
\]

The preferences of the family for consumption over time are represented by the utility integral:

\[
U_t = \int_{s}^{\infty} u(c_t) \exp[-\theta(t - s)] \, dt.
\]

The family's welfare at time \(s\), \(U_s\), is the discounted sum of instantaneous utilities \(u(c_t)\). The function \(u(\cdot)\) is known as the instantaneous utility function, or as "felicity"; \(u(\cdot)\) is nonnegative and a concave increasing function of the per capita consumption of family members. The parameter \(\theta\) is the rate of time preference, or the subjective discount rate, which is assumed to be strictly positive.\(^4\)

The Command Optimum

Suppose that a central planner wants at time \(t = 0\) to maximize family welfare. The only choice that has to be made at each moment of time is how much the representative family should consume and how much it should add to the capital stock to provide consumption in the future. The planner has to find the solution to the following problem:

\[
\max U_0 = \int_{0}^{\infty} u(c_t) \exp(-\theta t) \, dt
\]

subject to (2) and the constraints

\[k_0 \text{ given}; \quad k_t, c_t \geq 0 \text{ for all } t.\]

We characterize the solution using the maximum principle.\(^5\) The optimal solution is obtained by setting up the present value Hamiltonian function:

\[
H_t = u(c_t) \exp(-\theta t) + \mu_t [f(k_t) - nk_t - c_t].
\]

The variable \(\mu_t\) is called the costate variable associated with the state variable \(k\); equivalently it is the multiplier on the constraint (2). The value of \(\mu_t\) is the marginal value as of time zero of an additional unit of capital at time \(t\).

It is often more convenient to work, instead, with the marginal value, as of time \(t\), of an additional unit of capital at time \(t\), \(\lambda_t \equiv \mu_t \exp(\theta t)\); we shall do so here. Replacing \(\mu_t\) by \(\lambda_t\) in (5) gives

\[
H_t = [u(c_t) + \lambda_t (f(k_t) - nk_t - c_t)] \exp(-\theta t).
\]

We do not explicitly impose the nonnegativity constraints on \(k\) and \(c\).

Necessary and sufficient conditions for a path to be optimal under the assumptions on the utility and production functions made here are that\(^6\)
Chapter 2

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\[
H_t = 0.
\]
\[
d\mu_t = -H_t.
\]
\[
\lim_{t \to \infty} k_t\mu_t = 0.
\]

Using the definition of \( H(\cdot) \) and replacing \( \mu_t/L \) by \( \lambda \), we get

\[
u'(c_t) = \lambda_t. \tag{6}
\]
\[
d\lambda_t = \lambda_t[\theta + n - f'(k_t)], \tag{7}
\]
\[
\lim_{t \to \infty} k_tu'(c_t) \exp(-\theta t) = 0. \tag{8}
\]

Equations (6) and (7) can be consolidated to remove the costate variable \( \lambda \), yielding

\[
\frac{d\nu'(c_t)}{dt} = \theta + n - f'(k_t), \tag{7'}
\]
or equivalently

\[
\left[ \frac{c_tu''(c_t)}{u'(c_t)} \right] \left( \frac{dc_t}{dt} \right) = \theta + n - f'(k_t).
\]

The expression \( cu''(c)/u'(c) \) will recur often in this book. It reflects the curvature of the utility function. More precisely, it is equal to the elasticity of marginal utility with respect to consumption. If utility is nearly linear and if marginal utility is nearly constant, then the elasticity is close to zero. This elasticity is itself closely related to the **instantaneous elasticity of substitution**. The elasticity of substitution between consumption at two points in time, \( t \) and \( s \), is given by

\[
\sigma(c_t) \equiv -\frac{u'(c_s)/u'(c_t)}{u'(c_s)/u'(c_t)} \frac{d(c_t/c_s)}{d[u'(c_s)/u'(c_t)]}.
\]

Taking the limit of that expression as \( s \) converges to \( t \) gives \( \sigma = -u'(c_t)/u''(c_t)c_t \), so that \( \sigma(c_t) \) is the inverse of the negative of the elasticity of marginal utility. When utility is nearly linear, the elasticity of substitution is very large. Using the definition of \( \sigma \), (7') can be rewritten as

\[
\frac{dc_t}{dt} = \sigma(c_t)[f'(k_t) - \theta - n]. \tag{7''}
\]

The key conditions are (7) [or (7') or (7'')] and (8). Equation (7) is the Euler equation, the differential equation describing a necessary condition that has to be satisfied on any optimal path. It is the continuous time analogue of the standard efficiency condition that the marginal rate of substitution be equal to the marginal rate of transformation, as we shall show shortly. The condition is also known as the Keynes-Ramsey rule. It was derived by Ramsey in his classic article, which includes a verbal explanation attributed to Keynes. We now develop an intuitive explanation of this repeatedly used condition.

The Keynes-Ramsey Rule

The easiest way to understand the Keynes-Ramsey rule is to think of time as being discrete and to consider the choice of the central planner in allocating consumption between time \( t \) and \( t + 1 \). If he decreases consumption at time \( t \) by \( dc_t \), the loss in utility at time \( t \) is equal to \( u'(c_t)dc_t \). This decrease in consumption at time \( t \), however, allows for more accumulation and thus more consumption at time \( t + 1 \): consumption per capita can be increased by \( (1 + n)^{-1}[1 + f'(k_t)]dc_t \), leading to an increase in utility at \( t + 1 \) of \( (1 + n)^{-1}[1 + f'(k_t)]u'(c_{t+1})dc_t \). Along the optimal path small reallocations in consumption must leave welfare unchanged so that the loss in utility at time \( t \) must be equal to the discounted increase in utility at time \( t + 1 \). Thus

\[
u'(c_t) = (1 + \theta)^{-1}(1 + n)^{-1}[1 + f'(k_t)]u'(c_{t+1}).
\]

This condition can be rewritten as

\[
\frac{(1 + \theta)^{-1}u'(c_{t+1})}{u'(c_t)} = \frac{1 + n}{1 + f'(k_t)}, \tag{9}
\]

which states that the marginal rate of substitution (MRS) between consumption at times \( t \) and \( t + 1 \) is equal to the marginal rate of transformation (MRT), from production, between consumption at times \( t \) and \( t + 1 \). If the period is short enough, this condition reduces to equation (7').

A more rigorous argument runs as follows: Consider two points in time, \( t \) and \( s \), \( s > t \). We now imagine reallocating consumption from a small interval following \( t \) to an interval of the same length following \( s \). Decrease \( c_t \) by amount \( \Delta c_t \) at time \( t \) for a period of length \( \Delta t \), thus increasing capital accumulation by \( \Delta c_t\Delta t \). That capital is allowed to accumulate between \( t + \Delta t \) and \( s \), with consumption over that interval unchanged from its
original value. All the increased capital is consumed during an interval of length $\Delta t$ starting at $s$, with consumption thereafter being unchanged from the level on the original path. This variation from the optimal path is illustrated in figure 2.1.

For sufficiently small $\Delta c$ and $\Delta t$, such a reallocation should have no effect on welfare, provided the path is optimal. Thus

$$u'(c_s)\Delta c \Delta t + u'(c_s) \exp[-\theta(s - t)]\Delta c \Delta t = 0.$$ 

The relation between $\Delta c$ and $\Delta c_s$ is implied by

$$\Delta c \Delta t = \Delta k_s, \quad \Delta c_s \Delta t = \Delta k_s,$$

and

$$\Delta k_s = -\Delta k_s \exp \left\{ \int_{t+\Delta t}^{s} [f'(k_s) - n] \, dp \right\}.$$ 

Capital accumulated in the first interval $\Delta t$ grows at the rate $f'(k) - n$ between $t + \Delta t$ and $s$.

Eliminating $\Delta c$'s and $\Delta k$'s from the preceding relations gives

$$\frac{u'(c_s)}{u'(c_s) \exp[-\theta(s - t)]} = \exp \left\{ \int_{t+\Delta t}^{s} [f'(k_s) - n] \, dp \right\}. \quad (10)$$

Equation (10) has the same interpretation as equation (9), namely, that marginal rates of substitution and transformation are equal.

As this equality must hold for all $t$ and $s$, it follows that

$$\lim_{s \to t} \frac{dMRS(t, s)}{ds} = \lim_{s \to t} \frac{dMRT(t, s)}{ds}. \quad (11)$$

Applying this to (10) gives equation (7').

The Keynes-Ramsey rule, in discrete or continuous time, implies that consumption increases, remains constant, or decreases depending on whether the marginal product of capital (net of population growth) exceeds, is equal to, or is less than the rate of time preference. This rule is quite fundamental and quite intuitive: the higher the marginal product of capital relative to the rate of time preference, the more it pays to depress the current level of consumption in order to enjoy higher consumption later. Thus, if initially the marginal product of capital is high, consumption will be increasing over time on the optimal path. Equation (7') shows the specific role of the elasticity of substitution in this condition: the larger this elasticity, the easier it is, in terms of utility, to forgo current consumption in order to increase consumption later, and thus the larger the rate of change of consumption for a given value of the excess of the marginal product over the subjective discount rate.

The Transversality Condition

Equation (8), the transversality condition, is best understood by considering the same maximization problem with the infinite horizon replaced by a finite horizon $T$. In this case, if $u'(c_T) \exp(-\theta T)$ were positive (i.e., if the present value of the marginal utility of terminal consumption were positive), it would not be optimal to end up at time $T$ with a positive capital stock because it could, instead, be consumed. The condition would be

$$k_T u'(c_T) \exp(-\theta T) = 0.$$ 

The infinite horizon transversality condition (TVC) can be thought of as the limit of this condition as $T$ becomes large.

Two Useful Special Cases

CRRA

Two instantaneous utility functions are frequently used in intertemporal optimizing models. The first is the constant elasticity of substitution, or isoelastic, function.
In C, for $y = 1$. The basic economic property of this function is implied by its name. The elasticity of substitution between consumption at any two points in time, $t$ and $s$, is constant and equal to $(1/y)$. Thus, in equation (7”), $\sigma$ is no longer a function of consumption. The elasticity of marginal utility is equal to $-\gamma$.

When this instantaneous utility function is used to describe attitudes toward risk, something we shall do later in the book when we allow for uncertainty, $\gamma$ has an alternative interpretation. It is then also the coefficient of relative risk aversion, defined as $-u''(c)/u'(c)$. Thus this function is also called the constant relative risk aversion (CRRA) utility function.

Substantial empirical work has been devoted to estimating $\sigma$ under the assumption that it is indeed constant, by looking at how willing consumers are to shift consumption across time in response to changes in interest rates. Estimates of $\sigma$ vary substantially but usually lie around or below unity: the bulk of the empirical evidence suggests a relatively low value of the elasticity of substitution.

The second often used class of utility functions is the exponential, or constant absolute risk aversion (CARA), of the form

$$u(c) = \frac{e^{\alpha c}}{1 - \alpha}, \quad \alpha > 0.$$ 

Under this specification the elasticity of marginal utility is equal to $-\alpha$, and the instantaneous elasticity of substitution is equal to $(\alpha c)^{-1}$; thus $\sigma$ is decreasing in the level of consumption.

When interpreted as describing attitudes toward risk, this function implies constant absolute risk aversion, with $\alpha$ being the coefficient of absolute risk aversion, $-u''(c)/u'(c)$. Constant absolute risk aversion is usually thought of as a less plausible description of risk aversion than constant relative risk aversion; the CARA specification is, however, sometimes analytically more convenient than the CRRA specification and thus also belongs to the standard tool kit.

For the CARA utility function, the Euler equation becomes

$$\frac{dc}{dt} = \alpha^{-1}[f(k) - n - \theta].$$

In this case the change in consumption is proportional to the excess of the marginal product of capital (net of population growth) over the discount rate.

**Steady State and Dynamics**

The optimal path is characterized by equations (7”), (8), and the constraint (2). We start with the steady state. In steady state both the per capita capital stock, $k$, and the level of consumption per capita, $c$, are constant. We denote the steady state values of these variables by $k^*$ and $c^*$, respectively.

**The Modified Golden Rule**

From (7), with $dc/dt$ equal to zero, we have the modified golden rule relationship:

$$f'(k^*) = \theta + n. \quad (11)$$

The marginal product of capital in steady state is equal to the sum of the rate of time preference and the growth rate of population. Corresponding to the optimal capital stock $k^*$ is the steady state level of consumption, implied by (2):

$$c^* = f(k^*) - nk^*. \quad (12)$$

The golden rule itself is the condition $f(k) = n$: this is the condition on the capital stock that maximizes steady state consumption per capita. The modification in (11) is that the capital stock is reduced below the golden rule level by an amount that depends on the rate of time preference. Even though society or the family could consume more in a steady state with the golden rule capital stock, the impatience reflected in the rate of time preference means that it is not optimal to reduce current consumption in order to reach the higher golden rule consumption level.

The modified golden rule condition is a very powerful one: it implies that ultimately the productivity of capital, and thus the real interest rate, is determined by the rate of time preference and $n$. Tastes and population growth determine the real interest rate $(\theta + n)$, and technology then determines the capital stock and level of consumption consistent with that interest rate. Later in the chapter we will explore the sensitivity of the modified golden rule result to the formulation of the utility function $u(-)$ in (1).

**Dynamics**

To study dynamics, we use the phase diagram in figure 2.2, drawn in $(k, c)$ space. All points in the positive orthant are feasible, except for points on
unique initial level of consumption. For instance, with initial capital stock \( k_0 \), the optimal initial level of consumption is \( c_0 \). Convergence of \( c \) and \( k \) to \( c^* \) and \( k^* \) is monotonic. Note that in this certainty model the central planner knows at time 0 what the level of consumption and the capital stock will be at every moment in the future.

Local Behavior around the Steady State

Linearization of the dynamic system (2) and (7') yields further insights into the dynamic behavior of the economy. Linearizing both equations in the neighborhood of the steady state gives

\[
\frac{dc}{dt} = -\beta(k - k^*), \quad \beta \equiv [-f''(k^*)c^*]\sigma(c^*) > 0. \tag{13}
\]

and

\[
\frac{dk}{dt} = [f'(k^*) - n](k - k^*) - (c - c^*) \\
= \theta(k - k^*) - (c - c^*). \tag{14}
\]

The solution to this system of linear differential equations is most easily found by reducing it to a single second-order equation in \( k \). Differentiating (13) with respect to time, and using (13) to substitute for \( dc/dt \), gives

\[
\frac{d^2k}{dt^2} - \theta \left( \frac{dk}{dt} \right) - \beta k = -\beta k^*. \tag{15}
\]

The roots of the characteristic equation associated with the second-order differential equation are \( \theta \pm (\sqrt{\theta^2 + 4\beta})/2 \). One root is positive and the other negative, implying the saddle point property: the presence of a positive root implies that for arbitrary initial conditions, the system explodes; for any given value of \( k_0 \), there is a unique value of \( dk/dt \) such that the system converges to the steady state (see appendix B).

Let \( \lambda \) be the negative, stable root. The solution for \( k \), such that, starting from \( k_0 \), the system converges to \( k^* \) is

\[
k_\lambda = k^* + (k_0 - k^*) \exp(\lambda t).
\]

The speed of convergence is thus given by \( |\lambda| \). In turn \( |\lambda| \) is an increasing function of \( f' \) and of \( \sigma \), and a decreasing function of \( \theta \). The higher the elasticity of substitution, the more willing people are to accept low consumption early on in exchange for higher consumption later and the faster capital accumulates and the economy converges to the steady state.
2.2 The Decentralized Economy

Suppose that the economy is decentralized rather than centrally planned. There are two factor markets, one for labor and one for capital services. The rental price of labor, the wage, is denoted \( w \); \( r \) is the rental price of capital. There is a debt market in which families can borrow and lend.

There are many identical families, each with a welfare function given by equation (3). Each family decides, at any point in time, how much labor and capital to rent to firms and how much to save or consume. They can save by either accumulating capital or lending to other families. Families are indifferent as to the composition of their wealth, so the interest rate on debt must be equal to the rental rate on capital.\(^{18}\)

There are many identical firms, each with the same technology as described by equation (2); firms rent the services of capital and labor to produce output.\(^{19}\) The constant returns assumption means that the number of firms is of no consequence, provided the firms behave competitively, taking the prices (the real wage and rental rate on capital) facing them as given.\(^{20}\)

Both families and firms have perfect foresight; that is, they know both current and future values of \( w \) and \( r \) and take them as given. (Under certainty, perfect foresight is the equivalent of rational expectations, an assumption we will discuss at length later.) More formally, let \( \{w_t, r_t\}, t = [0, \infty) \), be the sequence of wages and rental rates. Then, given this sequence, each family maximizes at any time \( s \)

\[
U_s = \int_{s}^{\infty} u(c_t) \exp(-\theta(t-s)) \, dt
\]

subject to the budget constraint,

\[
c_t + \frac{da_t}{dt} + na_t = w_t + r_t a_t, \quad \text{for all } t, \, k_0 \text{ given},
\]

where

\[
a_t \equiv k_t - b_t.
\]

Family wealth, or more precisely nonhuman wealth, is given by \( a_t \), which is equal to holdings of capital, \( k_t \), minus family debt, \( b_t \).

At any time \( t \), the family supplies both capital and labor services inelastically: capital is the result of previous decisions and is given at time \( t \); by assumption, labor is supplied inelastically. Thus the only decision the family has to make at each point in time is how much to consume or save.

Firms in turn maximize profits at each point in time. Since their technology is characterized by the production function (2), first-order conditions for profit maximization imply that

\[
f'(k_t) = r_t,
\]

\[
f'(k_t) = f'(k_t) = w_t.
\]

Consider an arbitrary path of wages and rental rates. This sequence will lead each family to choose a path of consumption and wealth accumulation. Given that private debt must always be equal to zero in the aggregate, wealth accumulation will determine capital accumulation. The path of capital will in turn imply a path of wages and rental rates. The equilibrium paths of wages and rental rates are defined as those paths that reproduce themselves given optimal decisions by firms and households. We now characterize the equilibrium path of the economy.\(^{21}\)

The No-Ponzi-Game Condition

In stating the maximization problem of a family, we have not imposed the constraint that family nonhuman wealth, which is given by \( a_t \), at time \( t \), be nonnegative. In the absence of any restrictions on borrowing, the solution to the maximization problem is then a trivial one. It is for the family to borrow sufficiently to maintain a level of consumption such that the marginal utility of consumption equals zero (or an infinite level of consumption if marginal utility is always positive) and to let the dynamic budget constraint determine the dynamic behavior of \( a_t \). From the budget constraint it follows that this path of consumption will lead to higher and higher levels of borrowing (negative \( a \)), borrowing being used to meet interest payments on the existing debt. Ultimately, net indebtedness per family member will be growing at rate \( r - n \).

We need therefore an additional condition that prevents families from choosing such a path, with an exploding debt relative to the size of the family. At the same time we do not want to impose a condition that rules out temporary indebtedness.\(^{22}\) A natural condition is to require that family debt not increase asymptotically faster than the interest rate:

\[
\lim_{t \to \infty} a_t \exp\left[-\int_{0}^{t} (r_t - n) \, dv\right] \geq 0.
\]

This condition is sometimes known as a no-Ponzi-game (NPG) condition.\(^{23}\)

Although (18) is stated as an inequality, it is clear that as long as marginal
utility is positive, families will not want to have increasing wealth forever at rate \( r - n \), and that the condition will hold as an equality. Thus in what follows we use the condition directly as an equality.

To see what the condition implies, let us first integrate the budget constraint from time 0 to some time \( T \). This gives

\[
\int_0^T c_t \exp \left[ \int_t^T (r_v - n) \, dv \right] \, dt + a_T = \int_0^T w_t \exp \left[ \int_t^T (r_v - n) \, dv \right] \, dt + a_0 \exp \left[ \int_0^T (r_v - n) \, dv \right].
\]

Multiplying both sides by \( \exp[-\int_0^T (r_v - n) \, dv] \), that is, discounting to time zero, letting \( T \) go to \( \infty \), and using the NPG condition, gives

\[
\int_0^\infty c_t \exp \left[ - \int_0^t (r_v - n) \, dv \right] \, dt = a_0 + h_0.
\]

where

\[
h_0 \equiv \int_0^\infty w_t \exp \left[ - \int_0^t (r_v - n) \, dv \right] \, dt.
\]

This condition implies that the present value of consumption is equal to total wealth, which is the sum of nonhuman wealth, \( a_0 \), and of human wealth, \( h_0 \), the present value of labor income. Thus condition (18) allows us to go from the dynamic budget constraint (16) to an intertemporal budget constraint.\(^{24}\)

The Decentralized Equilibrium

Maximization of (3) subject to (16) and (18), carried out by setting up a Hamiltonian, implies the following necessary and sufficient conditions:

\[
du'(c_t)/dt = \theta + n - r_t,
\]

\[
limit \, a_t u'(c_t) \exp(-\theta t) = 0.
\]

In equilibrium, aggregate private debt \( b_m \) must always be equal to zero; though each family assumes it can freely borrow and lend, in equilibrium there is neither lending nor borrowing. Thus \( a_t = k_t \). Using this and equations (17) for \( w_t \) and \( r_t \), and replacing in (16) and (19) gives

\[
\frac{dc_t}{dt} + nk_t = f(k_t).
\]

Equations (20), (21), and (22) characterize the behavior of the decentralized economy. Note that they are identical to equations (8), (2), and (7') which characterize the behavior of the economy as chosen by a central planner. Thus the dynamic behavior of the decentralized economy will be the same as that of the centrally planned one. Our analysis of dynamics carries over to the decentralized economy.\(^{25}\)

The Role of Expectations

Equation (19), the Euler equation, gives the rate of change of consumption as a function of variables known at the current moment. It could be interpreted as suggesting that households need not form expectations of future variables in making their consumption/saving decisions and that the assumption of perfect foresight is not necessary. However, it is clear from the intertemporal budget constraint that the household cannot plan without knowing the entire path of both the wage and the interest rate. Expectations thus are crucial to the allocation of resources in the decentralized economy. In terms of the Euler equation, equation (19) only determines the rate of change, not the level of consumption.

Although it is difficult in general to solve explicitly for the level of consumption, this can be done easily when the utility function is of the CRRA family. In this case equation (19) gives

\[
\frac{dc_t}{dt} = \sigma(r_t - n - \theta).
\]

For a given value of initial consumption \( c_0 \), we can integrate this equation forward to get

\[
c_t = c_0 \exp \left[ \int_0^t \sigma(r_v - n - \theta) \, dv \right].
\]

Replacing in the intertemporal budget constraint gives the value of \( c_0 \) consistent with the Euler equation and the budget constraint

\[
c_0 = \beta_0(a_0 + h_0).
\]
where
\[
\beta_0^{-1} = \left[ \int_0^\infty \exp \left\{ \int_0^t \left[ (\sigma - 1)(r_v - n) - \theta \sigma \right] dv \right\} dt \right].
\]

Consumption is a linear function of wealth, human and nonhuman. The parameter \( \beta_0 \) is the propensity to consume out of wealth. It is generally a function of the expected path of interest rates. An increase in interest rates, given wealth, has two effects. The first is to make consumption later more attractive: this is the substitution effect. The second is to allow for higher consumption now and later: this is the income effect. In general, the net effect on the marginal propensity to consume is ambiguous. For the logarithmic utility function, however, \( \sigma = 1 \), and the two effects cancel; the propensity to consume is then exactly equal to the rate of time preference, \( \theta \), and is independent of the path of interest rates.

In general, expectations of interest rates affect both the marginal propensity to consume out of wealth and the value of wealth itself, through \( h_0 \). Expectations of wages also affect \( c_0 \) through \( h_0 \). Given these expectations, families decide how much to consume and save. This in turn determines capital accumulation and the sequence of factor prices.

What happens if expectations are incorrect? Agents will choose a different plan from our hypothetical central planner. When the divergence between actual and expected events causes them to revise their expectations, they will choose a new path that is optimal given their expectations. To pursue this line, we would have to specify how expectations are formed and revised. We defer that for later treatment.

### 2.3 The Government in the Decentralized Economy

In this section we introduce the government into the model. We assume that the government's spending requirements are fixed exogenously, and we examine the effects on the economy's equilibrium of, first, changes in the level of government spending and, second, different ways of financing a given level of government spending—either through taxation or borrowing.

Balanced Budget Changes in Government Spending

Suppose that a government is consuming resources and paying for them with taxes. The government's per capita demand for resources \( g_t \) is exogenous and, further, does not directly affect the marginal utility of consump-

\[ c_t + \frac{da_t}{dt} + na_t = w_t + r_t a_t - r_t \] \( a_t = k_t - b_{mt} \)

which, using the NPG condition, integrates to
\[ \int_0^\infty c_t R_t dt = k_0 - b_{p0} + \int_0^\infty w_t R_t dt - \int_0^\infty \tau_t R_t dt \] or equivalently,
\[ \int_0^\infty c_t R_t dt = k_0 - b_{p0} + h_0 - G_0. \] (23)

where \( R_t = \exp \left[ -\int_0^t (r_v - n) dv \right] \) is the factor by which future spending is discounted to the present and \( G_0 \) is the present discounted value of government spending, which is equal, by virtue of the assumption that \( \tau_t = g_t \), to the present discounted value of lump-sum taxes.

Government spending enters the intertemporal budget constraint, affecting the decisions of the family, the real equilibrium of the economy, and thus the time paths of \( w_t \) and \( r_t \) (and hence \( R_t \)). Suppose that the government demands a constant amount of resources, \( g \), per capita, where \( g \) is small. Using the equivalence between the decentralized and the centrally planned economy, we draw figure 2.3 to show the dynamics. The diagram is the same as figure 2.2, except that the output available for the private sector is reduced by the uniform amount \( g \), accounting for the vertical shift downward \( \delta \) the \( dk/dt = 0 \) locus to \( \delta (dk/dt = 0) \).

There is no equilibrium at low levels of the capital stock. But once there is sufficient capital to produce goods for the government, beyond \( k^* \), the analysis is similar to that in figure 2.2. The economy will proceed to a steady state at \( E' \) with the modified golden rule capital stock, and with consumption \( c^* \) smaller by an amount \( g \) than it was in the steady state in figure 2.2. In steady state government spending completely crowds out private consumption but has no effect on the capital stock.

Does a change in government spending have dynamic effects on capital accumulation? If the economy is in steady state initially, the change in government spending is reflected instantaneously in consumption with no dynamic effect on capital accumulation. If the economy is not initially in
Figure 2.3
The effects of an increase in public spending

steady state, whether or not the change in spending has a transitory effect depends on the characteristics of the felicity function. If, for example, felicity belongs to the CAR4 class, there is no dynamic effect on capital accumulation.

Debt Financing

Instead of financing itself through taxes with \( r_\pi = g_r \), the government may borrow from the private sector. Government debt must pay the same rate as capital, if agents are to hold it in their portfolios. Let \( b_r \) be per capita government debt. The government faces the following dynamic budget constraint:

\[
\frac{db_r}{dt} + nb_r = g_r - \tau_r + r_r b_r. 
\]

The left-hand side is government borrowing per capita, which is equal to the increase in the per capita debt \( (db_r/dt) \) plus the amount of debt \( (nb_r) \) that can, as a result of the growing population, be floated without increasing the amount of debt per capita. The right-hand side is the excess of government outlays, consisting of its purchases of goods and services and interest payments, over its tax receipts. The flow constraint says only that the government has to borrow when its outlays exceed its tax receipts, or that it repays debt or lends to the private sector when tax receipts exceed outlays.

Integrating this budget constraint and imposing the NPG condition this time on the government (that debt not increase faster asymptotically than the interest rate) gives an intertemporal budget constraint for the government:

\[
b_0 + \int_0^\infty g_r R_t \, dt = \int_0^\infty \tau_r R_t \, dt. 
\]  

The present value of taxes must be equal to the present value of government spending plus the value of the initial government debt \( b_0 \), given the NPG condition. Equivalently, the government must choose a path of spending and taxes such that the present value of \( g_r - \tau_r \), which is sometimes referred to as the primary deficit, equals the negative of initial debt, \( b_0 \): if the government has positive outstanding debt, it must anticipate running primary surpluses at some point in the future. For instance, it is consistent with (24) that the government maintain the initial value of the per capita debt, \( b_0 \), forever, running a primary surplus just large enough to pay the interest net of the amount of debt that can be financed by selling \( b_0 \) to each newborn person.

The presence of government debt also modifies the dynamic budget constraint of the family, which becomes

\[
c_t + \frac{da_t}{dt} + na_t = w_t + r_r a_t - \tau_r, 
\]  

with \( a_t \) now equal to \( k_t - b_{pt} + b_r \). Note that there is an implicit assumption in (25) that the family can borrow and lend at the same interest rate, \( r_r \), as the government.

Integrating this budget constraint subject to the NPG condition gives the following intertemporal budget constraint:

\[
\int_0^\infty c_t R_t \, dt = k_0 - b_{p0} + b_0 + \int_0^\infty w_t R_t \, dt - \int_0^\infty \tau_t R_t \, dt. 
\]  

The present value of consumption must be equal to the sum of nonhuman wealth, which is the sum of \( k_0 - b_{p0} \) and \( b_0 \), and of human wealth, which is the present value of wages minus taxes.

The government budget constraint shows that for a given pattern of government spending (and given \( b_0 \), the government has to levy taxes of a given present value; equivalently, the government need not run a balanced budget at every moment of time. For instance, starting from a balanced
budget, it can reduce taxes at some point, borrow from the public, and raise
future taxes to repay the interest and the debt.

What then is the effect of a change in the timing pattern of the taxes
raised to finance a given pattern of government expenditures? The answer
is given by replacing the intertemporal budget constraint of the government
in (26). This gives

$$\int_0^\infty c_t R_t dt = k_0 - b_{po} + \int_0^\infty w_t R_t dt - \int_0^\infty g_t R_t dt + b_{po}.$$  \hspace{1cm} (27)

Equation (27) is exactly the same as equation (23). Neither taxes nor
government debt appear in the budget constraint of the family. Only
government spending matters. This has a strong implication: for a given path
of government spending, the method of finance, through lump-sum taxation or
deficit finance, has no effect on the allocation of resources.

The intuition for this result is obtained by looking at the intertemporal
budget constraints of the government and families. A decrease in taxes, and
thus a larger deficit today, must according to the government budget
constraint lead to an increase in taxes later. According to the family budget
constraint, the current decrease and the anticipated future increase exactly
offset each other in present value, leaving the budget constraint unaffected.
Families thus do not modify their paths of consumption. They willingly save
the increase in current income, exactly offsetting the dissaving of the
government.

This conclusion is remarkable, for it provides one instance in which, so
long as the government ultimately meets its NPG condition, the size of the
national debt is of no consequence, and neither is deficit finance. We will
return several times to the issue of the effects of the national debt and deficit
finance and study the robustness of this strong neutrality result.

Distortionary Taxation of Capital

Distortionary taxation certainly affects the allocation of resources. Suppose
that the government taxes the return to capital at the rate $\tau_k$, and remits the
proceeds in lump-sum fashion to the private sector. If $r_t$ is the pre-tax rate
of return on capital, $(1 - \tau_k)r_t$ is the aftertax return on capital and must also
be the rate of return on private debt as capital and debt are perfect substi-
tutes in the family’s portfolio. The family’s flow budget constraint is now

$$c_t + \frac{da_t}{dt} + n a_t = w_t + (1 - \tau_k)r_t a_t + z_t.$$  \hspace{1cm} (28)
absence of distortionary taxation. Equivalently, if the government instead subsidized capital using lump-sum taxation, it could increase the steady state capital stock and level of consumption, as long as the steady state capital stock was below the golden rule level.

### 2.4 Application: Investment and Saving in the Open Economy

In this section we extend the closed economy optimizing model to the open economy. The extension not only sheds light on the optimal and actual responses of an economy to shocks, such as a reduction in productivity, but it also provides further insights into investment and saving behavior.

We extend the original model in two directions. In the closed economy model used so far, there was no cost to installing capital; whatever was saved could be added to the capital stock at no cost, and investment was purely passive. We now introduce costs of installation. This will be seen to imply that there is now both a well-defined saving decision and a well-defined investment decision.

If we maintained the assumption that the economy was closed, interest rates would have to adjust so that saving would be equal to investment at all points in time, or equivalently so that the demand for goods, consumption plus investment, would be equal to the supply of goods. Instead, we open up the economy, allowing international trade in both goods and assets. It is then possible for saving and investment not to be equal at any moment of time: temporary imbalances, current account deficits, can be financed by foreign borrowing. In this way we show most clearly the separate dynamics of investment and saving.

As before, there is equivalence between the command optimum and the decentralized equilibrium. We describe the command optimum in the text, and demonstrate its equivalence to the decentralized equilibrium in appendix C.

#### The Command Optimum

The optimization problem is

$$\max U_0 = \int_0^\infty u(c_t) \exp(-\theta t) \, dt$$  \hspace{1cm} (29)

subject to

$$\frac{db_t}{dt} = c_t + i_t \left[ 1 + T\left(\frac{i_t}{k_t}\right) \right] + \theta b_t - f(k_t).$$  \hspace{1cm} (30)

Figure 2.5

Costs of installation for investment

$$\frac{dk_t}{dt} = i_t,$$

$$T(0) = 0,$$

$$T'(\cdot) > 0,$$

$$2T'(\cdot) + \frac{i_t}{kT'(\cdot)} > 0.$$  \hspace{1cm} (31)

All variables are in per capita terms, and population is assumed here to be constant; $b$ denotes per capita debt. The felicity and production functions $u(\cdot)$ and $f(\cdot)$ have the same properties as in earlier sections.

There are two changes from the previous analysis. First, there are now costs of installing investment goods. It takes $i_t[1 + T(\cdot)]$ units of output to increase the capital stock by $i_t$ units. The amount $T(\cdot)$ per unit of investment is used up in transforming goods into capital. The properties of $T(\cdot)$ make the installation cost function $(i_t/k_t)T(\cdot)$, shown in figure 2.5, nonnegative and convex, with a minimum value of zero when investment is equal to zero; both investment and disinvestment are costly. For simplicity, we assume that there is no depreciation.

The other difference is that the economy can now borrow and lend freely abroad at the constant world interest rate $\theta$. This implies the flow budget constraint (30): the change in foreign debt $(db_t/dt)$ is equal to spending (on consumption, investment, and interest payments) minus output. The change
in foreign debt is the current account deficit, so that (30) is equivalent to the statement that the current account deficit is equal to the excess of absorption over production.

There is a simple relation among the current account deficit, saving, and investment that will be useful later. A brief refresher in national income accounting identities and definitions may be useful at this point. The current account deficit is equal to the change in foreign debt, which is in turn equal to interest payments minus net exports of goods, the trade surplus (nx):

\[
\frac{db}{dt} = \theta b - nx.
\]

Per capita GDP and GNP are given by

\[
\text{GDP} = c + i[1 + T(\cdot)] + nx,
\]
\[
\text{GNP} = \text{GDP} - \theta b.
\]

Saving is equal to GNP minus consumption:

\[
s = \text{GNP} - c = i[1 + T(\cdot)] + nx - \theta b = i[1 + T(\cdot)] - \frac{db}{dt}
\]

so that

\[
\frac{db}{dt} = \text{current account deficit} = i[1 + T(\cdot)] - s.
\]

The maximization problem as now stated has a simple solution. It is again a Ponzi-like solution, but now on the part of the central planner vis-à-vis the rest of the world. The country should borrow until the marginal utility of consumption is equal to zero, and then borrow further to meet interest payments on its debt. It is unlikely that the lenders would be willing to continue lending if the country’s only means of paying off its debt were to borrow more. Accordingly, we impose the NPG condition:

\[
\lim_{t \to \infty} b_t \exp(-\theta t) = 0.
\] (32)

To solve the intertemporal problem, we set up the present value Hamiltonian:

\[
H_t = \left[ u(c_t) - \mu_t \left( c_t + i_t \left[ 1 + T\left( \frac{i_t}{k_t} \right) \right] + \theta b_t - f(k_t) \right) + \mu_t q_t k_t \right] \exp(-\theta t).
\] (33)

Consumption and Investment: Basic Infinite Horizon Models

The costate variables on the flow budget constraint (30) and the capital accumulation equation (31) are \(-\mu_t \exp(-\theta t)\) and \(\mu_t q_t \exp(-\theta t)\). Necessary and sufficient conditions for a maximum are

\[
u'(c_t) = \mu_t \quad \text{(from } H_t = 0),
\]

\[
1 + T\left( \frac{i_t}{k_t} \right) + \left( \frac{i_t}{k_t} \right)^T T\left( \frac{i_t}{k_t} \right) = q_t \quad \text{(from } H_t = 0),
\]

\[
\frac{d(-\mu_t \exp(-\theta t))}{dt} = \theta \mu_t \exp(-\theta t),
\]

\[
\frac{d(-\mu_t q_t \exp(-\theta t))}{dt} = -\left\{ \mu_t \left[ f'(k_t) + \left( \frac{i_t}{k_t} \right)^T T\left( \frac{i_t}{k_t} \right) \right] \right\} \exp(-\theta t),
\]

\[
\lim_{t \to \infty} \mu_t q_t k_t \exp(-\theta t) = 0,
\]

\[
\lim_{t \to \infty} \mu_t b_t \exp(-\theta t) = 0.
\]

Equations (36) and (37) are the Euler equations associated with \(b\) and \(k\), respectively. Equations (38) and (39) are the transversality conditions associated with \(b\) and \(k\), respectively. We are now ready to characterize the solution. We start with consumption.

Consumption

Carrying out the differentiation in (36), we obtain

\[
\frac{d\mu_t}{dt} = 0,
\]

which implies that \(\mu\) is constant. In turn, this implies from (34) that consumption is constant on the optimal path. That is precisely what should be expected given the findings in sections 2.1 and 2.2 on the effects of the relationship between the interest rate and the rate of time preference on the profile of the consumption path.

To obtain the level of consumption, we integrate the flow constraint (30) using condition (38), which yields

\[
\int_0^\infty c_t \exp(-\theta t) dt = \int_0^\infty \left\{ f(k_t) - i_t \left[ 1 + T\left( \frac{i_t}{k_t} \right) \right] \right\} \exp(-\theta t) dt - b_0
\]

\[
= n_0.
\] (41)
The present discounted value of consumption is equal to net wealth at time 0, \( v_0 \), the present discounted value of net output (the contents of the braces in (41)) minus the initial level of debt. Since consumption is constant, (41) implies that
\[ c_t = c_0 = \theta v_0. \] (42)

**Investment**

Equation (35) contains a very strong result, namely, that the rate of investment (relative to the capital stock) is a function only of \( q_t \), which is the shadow price in terms of consumption goods of a unit of installed capital. Equation (35) implies a relation \( q = \Psi(i/k) \), with \( \Psi' > 0 \) and \( \Psi(0) = 1 \). Thus we can define an inverse function \( \varphi(\cdot) \) such that \( i/k = \varphi(q) \). From the properties of \( \Psi(\cdot) \), it follows that \( \varphi' > 0 \) and \( \varphi(1) = 0 \). Replacing in (31) gives
\[ \frac{dk}{dt} = i_t = k_t \varphi(q_t), \quad \varphi'(q) > 0, \varphi(1) = 0. \] (43)

Investment is, from (43), an increasing function of \( q \), the shadow price of capital. At the margin the planner equates the value of an addition to the capital stock with its marginal cost, which rises with the rate of investment. It makes sense to incur the higher marginal cost of investing faster only when the shadow value of capital is higher. Note that the rate of investment is zero when \( q = 1 \), when the shadow price of capital is the same as that of goods "on the hoof," so that positive rates of investment require \( q > 1 \). Note finally that the level of \( q \) determines the rate of investment relative to the capital stock, \( i_t/k_t \).

What in turn determines \( q_t \)? From (37), given (40),
\[ \frac{dq_t}{dt} = \theta q_t - f'(k_t) - \varphi(q_t)^2 T[\varphi(q_t)]. \] (37')

Integrating (37') subject to (39),
\[ q_t = \int_t^\infty \left[ f'(k_v) + \varphi(q_v)^2 T[\varphi(q_v)] \right] \exp[-\theta(v - t)] dv. \] (44)

The shadow price of capital is equal to the present discounted value of future marginal products. Marginal product is itself the sum of two terms: the first is the marginal product of capital in production; the second is the reduction in the marginal cost of installing a given flow of investment due to the increase in the capital stock (because the installation cost depends on the ratio of investment to capital). The higher the current or future expected marginal products or the lower the discount rate, the higher are \( q \) and the rate of investment. The most significant feature of (44) is that \( q \) and thus the rate of investment, does not depend at all on the characteristics of the utility function or on the level of debt. The investment decision is independent of the saving or consumption decisions in this open economy framework with an exogenous real interest rate.

**Saving, Investment and the Current Account**

Saving is given by
\[ s_t = f(k_t) - c_t - \theta b_t. \]

From the derivation of consumption above, \( c_t = \theta b_t \), so that
\[ s_t = f(k_t) - \theta \int_t^\infty \left( f(k_v) - f(k_v) \left[ 1 + T \left( \frac{1}{k_v^*} \right) \right] \right) \exp[-\theta(v - t)] dv. \] (45)

Thus saving is high when output is high compared to future expected output. The other distinctive result is that saving is independent of the level of debt: the equality of the marginal propensity to consume and of the interest rate implies that a higher level of debt leads to equal decreases in income and consumption, leaving saving unaffected.

Since the current account surplus is equal to saving minus investment, neither of which is affected by the stock of debt, the current account is also independent of the stock of debt.

**Steady State and Dynamics**

The dynamic system characterizing the behavior of the economy is recursive, with (43) and (37') determining investment, capital, and output. The level of consumption and debt dynamics are then determined by (42) and (30).

**Investment and Capital**

In steady state \( dk/dt = dq/dt = 0 \). Accordingly, from (31), from \( \varphi(1) = 0 \), and from (37'),
\[ q^* = 1, \quad f'(k^*) = 0. \] (46)

where the asterisks denote steady state values.
In steady state the rate of investment is zero. The shadow price of capital must therefore be equal to its replacement cost, or \( q = 1 \); in turn, the marginal product of capital has to be equal to the interest rate, which is itself equal to the rate of time preference. We limit our analysis of the dynamics of investment and capital to a neighborhood around the steady state. To do so, we linearize (43) and (37') around \( q' = 1 \) and \( k^* \):

\[
\begin{bmatrix}
\frac{dk}{dt} \\
\frac{dq}{dt}
\end{bmatrix} = \begin{bmatrix}
0 & k^*\varphi'(1) \\
-f''(k^*) & 0
\end{bmatrix} \begin{bmatrix} k - k^* \\ q - 1 \end{bmatrix}.
\]  

(47)

Figure 2.6 gives the phase diagram corresponding to (47). The \( \frac{dk}{dt} = 0 \) locus is horizontal at \( q = 1 \); the \( \frac{dq}{dt} = 0 \) locus is downward sloping. The arrows indicate directions of motion. There is therefore a unique path converging to the steady state, the downward-sloping path SS.

The dynamics of investment are implied by the saddle point path SS. Given an initial capital stock \( k_0 \), the initial value of \( q, q_0 \) is read off SS, and the associated level of investment follows from (43). Since \( q_0 \) in this case exceeds unity, capital accumulates over time. Output increases and so does net output, which is equal to \( f(k) - i[1 + T(i/k)] \); output increases while investment decreases over time.

Consumption and Debt

We have already seen that the level of consumption is constant, determined by the path of net output (which is itself determined by the path of investment) and by the initial stock of debt. Figure 2.7 shows a path of net output that increases over time as the capital stock increases to its steady state level. We will now determine the level of consumption in figure 2.7. Assume that the initial stock of debt \( b_0 \) was zero. The constant level of consumption must then, from (41), be such that the present discounted value of net output minus consumption is zero, or equivalently that the present discounted value of current and future trade surpluses is zero. Graphically, the discounted values of the two hatched areas in figure 2.7 must be equal and opposite in sign; the level of consumption is determined by drawing a horizontal line such that the two areas are equal in present value.

In figure 2.7 net output increases over time. Net output accordingly starts out below and eventually exceeds consumption. The initial excess of consumption over net output is achieved by foreign borrowing, or by running a current account deficit. Debt accumulates during this phase. Eventually, net output rises sufficiently so that the trade balance shows a surplus. In steady state the current account must be balanced. The trade surplus is offset by interest payments on debt. The steady state level of debt \( b^* \) is positive and such that \( \theta b^* \) is equal to \( AB \), the trade surplus, in figure 2.7. The presence
of the debt reflects the decision to consume at a rate above the level of net output early in time.

Productivity Shocks and the Current Account

The paths of consumption and the current account in the preceding analysis can serve as a baseline for the analysis of the effects of shocks to productivity on the economy. Suppose that output is given by

\[ y = (1 - z_0)f(k) - z_1. \]

Here \( z_0 \) is a multiplicative shock, and \( z_1 \) is an additive shock. Reflecting the experience of oil shocks in the 1970s, we take the shocks to be adverse, and consider increases in either \( z_0 \) or \( z_1 \) that reduce output given the capital stock.\(^4\)

A Permanent Additive Shock

We start by considering an unexpected permanent increase in \( z_1 \), from a value of zero, with the economy initially in steady state.\(^4\) An increase in \( z_1 \) has no effect on the marginal product of capital and thus no effect on investment and the capital stock. Since the change is both unexpected and permanent, the increase in \( z_1 \) leads to an unexpected and permanent reduction in net output by the same amount, \( z_1 \). From (41) it follows that consumption falls by exactly the same amount. Saving therefore remains unchanged. Further, with both savings and investment unchanged, the current account is unaffected by the productivity shock.

In this case of an unexpected permanent reduction in \( z_1 \), the economy takes its losses immediately, and with no further consequences for the allocation of resources.\(^4\)

A Transitory Additive Shock

Suppose now that \( z_1 \) increases unexpectedly but temporarily at time 0 for a period of length \( T \). There is still no effect on investment, but there will now be a change in saving and the current account. The change in the present discounted value of net output is given by

\[ -z_1 \theta^{-1}[1 - \exp (-\theta T)] \]

so that the change in consumption is given by \( -z_1[1 - \exp (-\theta T)] \).

This change in consumption is permanent. If \( T \) is small, the change in consumption is also small. Agents cut consumption only a little, and most of the decrease in output translates into a reduction in saving and a current account deficit. After output returns to normal, the economy runs a permanent trade surplus to pay for the increased interest payments on the debt. If \( T \) is large, the change in consumption is larger, the reduction in saving and the increase in debt smaller. As \( T \) tends to infinity, we obtain the same results as in the permanent case.

A Permanent Multiplicative Shock

Finally, let \( z_0 \) increase from zero to a positive value at \( t = 0 \). Because the marginal product of capital is \( (1 - z_0)f'(k) \), a change in \( z_0 \) affects investment. We start by analyzing the effects of the change on investment and output.

As figure 2.8 shows, the increase in \( z_0 \) shifts the \( dq/dt = 0 \) locus to the left. The \( dk/dt = 0 \) locus is unaffected. The steady state of the economy shifts from \( E \) to \( E' \). At \( E' \) the steady state capital stock \( k^* \) is lower than \( k^* \), the initial steady state capital stock; the marginal product of capital \((1 - z_0)f'(k)\) is again equal to \( \theta \).

The new saddle point path is \( SS' \). With the initial capital stock given by \( k^* \), the path of adjustment is composed of a jump at time 0 from \( E \) to \( A \), and a movement over time from \( A \) to \( E' \). The rate of investment is negative on
the adjustment path, returning to zero as the economy moves to the new lower steady state capital stock.

Net output, which is equal to \((1 - z_0)f(k) - i[1 + T(i/k)]\), may either increase or decrease initially, depending on whether the fall in output when \(z_0\) increases is larger or smaller than the decline in investment. In the long run, however, the effect is unambiguous. As investment returns to zero and the capital stock falls to a lower level, net output must be lower in the new steady state, because the initial effect of the adverse shock is compounded by lower capital accumulation.

In figure 2.9 we show net output falling initially, and then declining further to its new steady state level. The new level of consumption is determined again by the condition that the present value of the hatched areas above and below it be equal.

With net output above consumption immediately after the shock, the economy is saving in anticipation of lower net output later. In figure 2.9 the economy runs current account and trade surpluses immediately after the shock and becomes a net owner of foreign assets in steady state.

These examples show that there is no simple relation between adverse supply shocks and the current account even in the simple model developed in this section. What happens depends on the nature of the shocks affecting the economy, for example, whether they are additive or multiplicative, temporary or permanent. Sachs (1981) has used a closely related model to study the effects of the oil shocks of the 1970s on the current accounts of different groups of countries. He argues that the response of the industrialized developing countries, which borrowed extensively abroad during that period, conforms roughly to the predictions of the model.

2.5 The Utility Function

The assumption that the utility function is additively separable, with exponential discounting, produces strong results. In particular, because the modified golden rule relation (11) fixes the steady state real interest rate, the model of this chapter implies that no policy changes or shocks to the production function can affect the steady state aftertax real interest rate.

There is, however, no strong reason, beyond analytical convenience, to assume additive separability or a constant rate of time preference. Marginal utility of current consumption may well depend on past consumption through habit or through boredom effects. One may well have a rate of time preference that changes through life or, say, between summers and winters. What happens when we allow for such complications? This is the question we briefly explore in this last section.

Relaxation of the assumption of additive separability may lead to far more complex dynamics. Assume, for instance, the following form of the felicity function at time \(t\):

\[ u(c_t, z_t), \]

where \(z_t\) depends on past rates of consumption. The assumption here is that the history of consumption affects the marginal utility of current consumption. Ryder and Heal (1973) have shown that, with only this modification to the standard Ramsey model, optimal paths may overshoot the modified golden rule steady state, and that oscillatory approaches to the steady state are possible.

In this section we continue, however, to assume that the felicity function takes the form \(u(c_t)\) and examine, instead, the assumption of a constant rate of time preference. We first show a further implication of constant rates of time preference and then explore the rationale for that formulation and present an alternative representation.

### Differences in Rates of Time Preference

We have assumed until now that all families have the same discount rate \(\theta\). There is no reason why this should generally be so. Consider the alternative in which there are \(m\) different types of families, ordered by decreasing impatience, with rates of time preference \(\theta_1 > \cdots > \theta_m > 0\). The economy...
is in other respects identical with that of section 2.2, except that for expository simplicity we assume zero population growth.

We now show that in steady state the interest rate $r$ must be equal to the lowest rate of time preference $\theta_m$. Suppose that this is not the case, and that $r$ is smaller than $\theta_m$. Then with $r$ smaller than all rates of time preference, all families have decreasing consumption over time, by equation (19); but if all families have decreasing consumption and there is no population growth, the economy cannot be in steady state with constant aggregate consumption.

Suppose, instead, that $r$ is greater than $\theta_m$. All families with discount rate smaller than $r$ have increasing consumption, and others have decreasing consumption. The share of total consumption accounted for by families with increasing consumption must be increasing over time. Indeed, the share of total consumption accounted for by the families with the lowest discount rate must eventually tend to one. Total consumption must therefore eventually be increasing; this is again inconsistent with being in steady state.

In steady state therefore the interest rate $r$ equals $\theta_m$. The consumption of the most patient consumers is constant. The consumption of all other families is declining so that eventually their share in total consumption is zero. Slow and steady wins the race, and all the wealth; not only do the most patient families own all physical capital, but they also “own” the human capital of others who pay over all their labor income in return for past borrowing.47 The model paints a somber, though unrealistic, picture of the dynamics of income and wealth distribution.

A slightly less extreme result is obtained if consumers are prohibited from borrowing against labor income and thus are constrained to have nonnegative financial wealth. The steady state will still have $r = \theta_m$. The most patient families will hold all nonhuman wealth; all others will have a level of consumption equal to their labor income.48 This result, though less drastic, is still not a good description of income distribution dynamics.49

There are many simplifications in this model, including the absence of uncertainty and the existence of infinitely lived families. These are possible directions in which to search for better models of income distribution dynamics. Another direction, to which we turn below, is the specification of preferences. What happens when we relax the assumption that the discount rate is constant?

Calendar Time, Time Distance, and Time Consistency

Suppose that, instead of the assumption of a constant rate of time preference, the utility integral is given by

$$U_t = \int_s^\infty u(c_s)D(t, t-s, x(t))\,dt.$$  \hspace{1cm} (48)

Utility is a weighted integral of felicities at different times, with the weighting function $D(\cdot \cdot \cdot \cdot \cdot)$, referred to as a discount function. In (48) we make the discount function potentially a function of calendar time, $t$, of time distance, $t - s$, and possibly other variables, $x(t)$, for instance, the rate of consumption itself. Note that the formulation (1) makes the discount factor between any two periods purely a function of time distance: the rate of discount applied to utility for any particular number of years (say, $T$) in the future is always the same [in this case $\exp(-\theta T)$].

In any optimal program the marginal rate of substitution between consumption at any two dates is equal to the marginal rate of transformation. Using (48),50 we now characterize the optimal program. Consider two planning dates, $t_1$ and $t_2$, and two points in time about which plans are made, $t_1$ and $t_2$; assume that $t_2 > t_1 > t_1 \geq t_2$.

As of planning date $t_1$, the marginal rate of substitution between consumption at time $t_1$ and consumption at time $t_2$ is given by

$$u'(c(t_1))D(t_1, t_1 - t_1) \quad u'(c(t_2))D(t_2, t_2 - t_1).$$ \hspace{1cm} (49)

Now consider the same marginal rate of substitution between consumption at time $t_1$ and consumption at time $t_2$ viewed as of planning date $t_2$,

$$u'(c(t_1))D(t_1, t_1 - t_2) \quad u'(c(t_2))D(t_2, t_2 - t_2).$$ \hspace{1cm} (50)

Comparison of (49) and (50) indicates that since $D(t, t - t_1)$ is generally different from $D(t, t - t_2)$, there is no reason for the rates of substitution between consumptions at times $t_1$ and $t_2$ to be the same from the two different planning dates. This implies that the optimal plan chosen at time $t_1$ will no longer be optimal as of time $t_2$. The optimal plan is therefore time inconsistent: the family’s optimal plan changes over time even though no new information becomes available.

There are now two issues: First, under what restrictions on the discount function $D(\cdot \cdot \cdot \cdot \cdot)$ is the optimal plan time consistent? And second, what happens when the optimal plan is time inconsistent?

For the optimal plan to be the same as of time $t_1$ and time $t_2$, the marginal rates of substitution (49) and (50) must be the same. This can happen if the
discount function $D(\cdot)$ is either an exponential function of time distance, $\exp[-\theta(t-s)]$ [check that this form ensures that the rates of substitution (49) and (50) are the same], or purely a function of calendar time or calendar values of other variables. The first instance gives one possible rationale for assuming exponential discounting.

Dealing with Inconsistency

What happens, though, if people do have a discount function that leads to time inconsistency? This question can be handled at many levels, but we do not dig deeply. One possibility is that of precommitment. Consumers, having solved their optimal plan at time $t=0$, may find a way of committing themselves to the plan to prevent what they then (at $t=0$) would regard as backsliding. For instance, they could commit themselves to a savings path by entering a savings plan.

Another possibility is that consumers recognize that their tastes will be changing and make their plans assuming that they will at each future moment follow their tastes of that moment. They then choose a consistent plan in the sense that all future actions are correctly taken into account in the planning process. If their discount function is only a function of time distance, this will lead them to act as if they had a constant rate of time preference through life.

This gives two possible rationales for assuming exponential discounting. The first is that exponential discounting leads to optimal programs that are time consistent. The other is that even if families do not exponentially discount the future but behave in time-consistent fashion, they may act as if they had exponential discounting. As we have seen, however, exponential discounting is not the only form of discounting that leads to time consistency. We now examine a formulation of the discount function in which the discount rate depends on the level of consumption. This formulation leads to a rate of time preference that changes through time but still implies time consistency of the optimal program.

Dependence of the Discount Rate on Utility

Uzawa (1968) considered the possibility that the rate of time preference depends on the level of utility or consumption. Uzawa's utility functional is

$$\int_0^\infty u(c_t) \exp \left\{ -\int_0^t \theta[u(c_s)] \, ds \right\} \, dt.$$  

The innovation is that the instantaneous rate of time preference is a function of the current level of utility, and thus of consumption. Uzawa specified that

$$\theta'(\cdot) > 0. \tag{51}$$

The implication of (51) is that a higher level of consumption at time $v$ increases the discount factor applied to utility at and after $v$. In steady state a higher level of consumption implies a higher rate of time preference. The assumption $\theta'(\cdot) > 0$ is difficult to defend a priori; indeed, we usually think it is the rich who are more likely to be patient. Assumption (51) is, however, needed for stability: if the rate of time preference fell with the level of consumption, the rich would become richer over time. That problem does not arise when, as in (51), the rate of time preference increases with the level of consumption.

We do not analyze the dynamics of this growth model but briefly characterize the steady state. Setting up the full model as in section 2.1, and deriving optimality conditions for the family, we obtain

$$[u'(c_t) - \lambda_t] - \frac{\theta[u(c_t)] u'(c_t)}{\theta[u(c_t)]} \{u(c_t) - \lambda_t [f(k_t) - n k_t - c_t]\} = 0.$$  

$$\frac{d\lambda_t}{dt} = \theta[u(c_t)] + n - f'(k_t). \tag{52}$$

$$\lim k_t \lambda_t \exp \left\{ -\int_0^t \theta[u(c_s)] \, ds \right\} = 0. \tag{53}$$

The first equation gives the relation between the costate variable and consumption. Note that the dependence of the discount rate considerably complicates this relation, which we shall not discuss further. Note also that if $\Theta(\cdot)$ is constant, this relation reduces to $u'(c) = \lambda$, as in the Ramsey model. Equation (52) gives the relation between the rate of change of the costate variable the discount rate, and the marginal product; this relation is as in the Ramsey model. Equation (53) is the standard transversality condition.

The other equation is the capital accumulation equation:

$$\frac{dk_t}{dt} = f(k_t) - nk_t - c_t.$$  

In steady state $\frac{dc}{dt} = \frac{dk}{dt} = 0$ so that

$$\theta[u(c^*)] = f'(k^*) - n,$$

$$c^* = f(k^*) - nk^*.$$  

These two loci are drawn in figure 2.10.
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Figure 2.10
Dynamics with endogenous time preference

The $\frac{dk}{dt} = 0$ locus is the same as in section 2.1. The $\frac{dz}{dt} = 0$ locus however, is now downward sloping rather than vertical. The saddle point steady state equilibrium is at point $E$.

Consider now an additive productivity shock, an increase in $z$, that shifts the $\frac{dk}{dt} = 0$ locus down uniformly. The new equilibrium is at $E'$. The steady state capital stock rises so that the reduction in consumption caused by lower productivity is compensated for by an increase in capital. The rate of time preference and the real interest rate are lower in the new equilibrium at $E'$ than they were at $E$.

The results in figure 2.10 contrast sharply with those that would occur in the model of section 2.1. In that case the fall in productivity would leave the steady state real interest rate unaffected and would result in a reduction in consumption exactly equal to the decrease in output.

Returning to the issue that motivated our look at the Uzawa formulation, consider the situation where families have different discount rate functions. In steady state all discount rates will be equal. This implies a distribution of consumption across families and an associated steady state distribution of wealth. Families with more patience, in the sense that at a given level of consumption their rate of time preference is lower, achieve higher steady state wealth and consumption.

This specification avoids the pathological results of the constant discount rate case. Nonetheless, the Uzawa function, with its assumption $\theta'(\cdot) > 0$, is not particularly attractive as a description of preferences and is not recommended for general use. A nondegenerate steady state, when individual tastes differ, can also be achieved by assuming that agents have finite lives; this is a more plausible avenue which we develop in chapter 3.

Appendix A: Ruling Out Explosive Paths in the Ramsey Model

To show that the saddle point path $DD$ in figure 2.2 is the optimal path, suppose that the initial capital stock is $k_0$, $0 < k_0 < k^*$. Consider any trajectory that starts above point $D$, at $D'$, say. This path implies that the economy reaches zero capital in finite time. The proof turns on the fact that on such a path $\frac{d^2k}{dt^2}$ eventually becomes negative. Differentiating (2) gives

$$\frac{d^2k}{dt^2} = f'(k) - n \left( \frac{dk}{dt} \right) - \frac{dc}{dt} < 0, \quad \text{as} \quad \frac{dc}{dt} > 0, \quad f'(k) - n > 0.$$

Thus $k_0 = k_0 + \frac{1}{n} (dk/dt) dt$ will reach zero in finite time.

Note that $c$ is rising on the path starting at $D'$ all the time until it hits the axis at point $B$. But when the path reaches $B$, $k$ is zero, and the economy has to move to the origin. Thus $c$ has to jump from a positive value to zero. But such a jump violates the necessary condition (7'), and it thus cannot have been optimal to start at $D'$.

Consider, alternatively, a trajectory starting below $D$, for example, at $D''$. This path converges asymptotically to $A$. But such a path violates the transversality condition. At points close to $A$, $k$ is approximately constant, whereas from (7') and $k > k^*$

$$\frac{du(c)}{dt} = 0 + n - f'(k) > 0.$$

Thus as $t$ tends to infinity and the trajectory approaches $A$, the transversality condition is violated.

Similar arguments apply if the initial capital stock is larger than $k^*$. It follows that the saddle point path $DD$ is the unique path that satisfies conditions (2), (7'), and (8).

Appendix B: Local Behavior of Capital around the Steady State in the Ramsey Model

The characteristic equation associated with equation (15) is

$$s^2 - \theta s - \beta = 0.$$

It has two roots:

$$s = \frac{-\theta \pm \sqrt{\theta^2 + 4\beta}}{2} < 0$$

and
Thus paths that satisfy equation (15) are given by
\[ k_i - k^* = c_0 \exp(i_0) + c_1 \exp(i_0), \]
where \( c_0 \) and \( c_1 \) are arbitrary constants.

As \( k_0 \) is given from history, \( c_0 \) and \( c_1 \) must satisfy
\[ k_0 - k^* = c_0 \exp(0) + c_1 \exp(0) = c_0 + c_1. \]
In addition, as \( \mu \) is positive, \( c_1 \) must be equal to zero for \( k \) to converge to \( k^* \). Thus \( c_1 = 0 \) and \( c_0 = k_0 - k^* \). This implies in turn that
\[ k_i = k^* + (k_0 - k^*) \exp(i_0). \]

### Appendix C: Command Optimum and Decentralized Equilibrium in the Open Economy Model

We show here the equivalence of the command optimum and the decentralized competitive equilibrium in the open economy model of section 2.4. For notational simplicity, we assume that there are as many firms as families so that the same symbol denotes the ratio of a variable per capita or per firm.

The structure of the economy is the following: Firms rent labor services in the labor market but own the capital stock: they finance investment through retained earnings. Families supply labor services and own the firms, receiving profits net of investment expenses. They allocate their income between consumption and saving, where saving takes the form of lending to the rest of the world.\(^5\)

### Value Maximization by Firms

For simplicity, we do not explicitly model the labor market. Labor is supplied inelastically so that labor market equilibrium implies that each firm hires one worker, paying wages of \( \{w_i\} \mid i = 0, \infty \). The decision problem of a representative firm at time zero is then to choose the time path of investment that maximizes the present discounted value of cash flows:

\[
\max V_0 = \int_0^\infty \left\{ f(k_i) - i_0 \left[ 1 + T\left( \frac{i}{i_0} \right) \right] - w_i \right\} \exp(-\theta t) dt
\]

subject to \( dk_i/dt = i \), and the same technology as the central planner.

By letting \( q_i \exp(-\theta t) \) be the Lagrange multiplier associated with the capital accumulation equation and setting up a present value Hamiltonian, the first-order conditions lead to equations identical to (43), (37'), and (44). Firms invest until the marginal cost of investment is equal to the shadow value of installed capital. \( q \). This shadow value is itself equal to the present discounted value of future marginal products. Firms choose the same path of investment and capital accumulation as the central planner.

### Utility Maximization by Families

Each family supplies one unit of labor inelastically, receiving wage \( w_i \) and dividends \( \pi_i \). Its only decision problem is to choose a path of consumption that maximizes

\[
U_0 = \int_0^\infty u(c_i) \exp(-\theta t) dt.
\]

It can borrow and lend on the world market at the rate \( \theta \). The dynamic budget constraint is therefore

\[
\frac{db_i}{dt} = c_i + \theta b_i - \pi_i - w_i.
\]

To this we add the NPG condition:

\[
\lim_{t \to \infty} \exp(-\theta t) = 0.
\]

The solution to this maximization problem is given by

\[
c_i = c_0 = 0 \int_0^\infty (\pi_i + w_i) \exp(-\theta t) dt.
\]

Replacing \( \pi_i \) by its value from (C2) in (C6) gives the same path of consumption as equation (42). Families will choose the same path of consumption as the central planner.

### Appendix D: Saddle Point Equilibrium in the Linearized \((k, q)\) System

Equation (47) linearizes the dynamic system that describes the behavior of \( q \) and \( k \) around the steady state values. The solution to such a linear system is given by

\[
k_i - k^* = c_{11} \exp(y_1 t) + c_{12} \exp(y_2 t),
\]

where \( y_1 \) and \( y_2 \) are the roots of the characteristic equation associated with (47), namely.

\[
\begin{vmatrix}
0 - y & k^* q'(1) \\
-f'(k^*) & 0 - y
\end{vmatrix} = 0.
\]
and where \( \{c_1, c_2\} \) and \( \{c_1, c_2\} \) are eigenvectors associated with each of these two roots.

The roots are given by

\[
y = \frac{\theta \pm \sqrt{\theta^2 - 4f'(k^*)k^*\psi'(1)}}{2}
\]

Both roots are real, with one root negative and the other positive. The positive root exceeds \( \theta \).

Denote the negative root by \( y_1 \). The eigenvector associated with \( y_1 \) is given by

\[
\begin{bmatrix}
-\frac{y_1}{k^*}\psi'(1) \\
-f'(k^*) & -y_1
\end{bmatrix}
\begin{bmatrix}
c_{11} \\
c_{21}
\end{bmatrix} = 0,
\]

so that \( c_{21} = \{y_1[k^*\psi'(1)]^{-1}\}c_{11} \). Examining (A7), we see that for the path that converges to \( (k^*, 1) \), both \( c_{12} \) and \( c_{22} \) must be equal to zero. (Zero is always an eigenvector.)

To calculate the constants \( c_{11} \) and \( c_{21} \), note that at time zero, the first row of \( (D1) \) is

\[
k_0 - k^* = c_{11}.
\]

Replacing the \( c \)'s by their values in \( (D1) \) gives the converging path for \( k \) and \( q \).

On all paths other than the converging path, \( c_{12} \) and/or \( c_{22} \) are different from zero. Thus \( q \) and \( k \) eventually increase at rate \( y_2 \). This implies that \( dk \) eventually increases at rate no less than \( y_2 \), which is itself greater than \( \theta \). Thus they all violate the transversality condition (39).

Of course, the proof that the transversality condition is violated on all but the saddle point path in the linearized system does not establish the fact that the paths of the original system that are not saddle point paths explode at a rate greater than \( \theta \). A complete proof requires a characterization of the dynamics of the original nonlinear system along the lines of the proof presented in appendix A.

### Problems

1. **The Solow growth model.** (This follows Solow 1956.)
   
   (a) Consider an economy with a population growth rate equal to \( n \), with constant returns to scale in production, and in which individuals save a constant fraction, \( s \), of their income. Show that the differential equation describing the behavior of the capital stock per capita is given by

   \[
   \frac{dk}{dt} = s(k) - nk.
   \]

   where \( f(k) \) is the production function per capita and \( s \) is the savings rate.

   (b) Characterize the steady state capital stock per capita in this model.

   (c) Examine the stability of the system, and characterize the adjustment of the capital stock toward its steady state.

   (d) Can a constant saving rate along the path of adjustment be consistent with intertemporal utility maximization by infinitely long-lived individuals?

   (e) Assume that factor markets are competitive. Show that the savings rate that leads to the golden rule capital stock is equal to the share of capital in production. Explain.

2. **Growth with exogenous technological progress.**

   Suppose that, in a Ramsey economy, production is given (as in note 13) by the function

   \[
   y_t = f(k_t, \exp(\phi t)N_t),
   \]

   where \( \phi \) is the constant and exogenous rate of technical progress. Assume that the population grows at rate \( n \) and that the utility function is of constant relative risk aversion form, with a coefficient of relative risk aversion equal to \( \gamma \).

   (a) Derive and interpret the modified golden rule condition in this case.

   (b) Characterize the dynamics of consumption and capital accumulation.

   (c) Suppose that the economy is in steady state and that \( \phi \) decreases permanently and unexpectedly. Describe the dynamic adjustment of the economy to this adverse supply shock.

3. **Optimal consumption with exponential utility.**

   Consider a family, growing at rate \( n \) and with discount rate \( \theta \), that faces a given path of future wages and interest rates and has a constant absolute risk aversion utility function, with a coefficient of risk aversion \( \alpha \). Solve for the path of consumption, as is done in the text for the CRRA utility function.

4. **Government spending in the Ramsey model.**

   (a) In the Ramsey model, suppose that the government unexpectedly increases government spending, raising it from a base level \( g_0 \) to the level \( g_1 \) (per capita in both cases), starting from steady state. Analyze the effects of this increase on the paths of consumption and capital accumulation.

   **Note:** You may want to use the equivalence between the command and market solutions and treat the increase in \( g \) as a negative additive productivity shock.

   (b) Do the same exercise, assuming that the economy is not initially in steady state.

   Characterize the dynamic effects when utility is of the CARA form. Explain.

   (c) Suppose, instead, that the increase in government spending is announced at time \( t_0 \) to take place at time \( t_1 \), with \( t_1 > t_0 \). Characterize the dynamic effects on consumption and capital accumulation from \( t_0 \).

   **Note:** Phase diagrams are convenient to use when characterizing the effects of such anticipated changes. Note that between \( t_0 \) and \( t_1 \), the equations of motion are given by the dynamic system with \( \gamma = \gamma_0 \), and that after \( t_1 \), the equations of motion are given by the dynamic system with \( \gamma = \gamma_1 \). Note further that \( \gamma \) cannot jump unexpectedly at time \( t_1 \). Finally that \( k \) at time \( t_0 \) is given and that the system must converge to the new equilibrium. Show that these conditions uniquely define the path of adjustment. (Abel 1981 characterizes the effects of anticipated or
temporary changes in taxation on investment within the $q$ theory using such phase diagrams.)

5. Savings and investment with costs of adjustment in a closed economy.
(This follows Abel and Blanchard 1983.)

Assume that there are costs of adjusting the capital stock, as in section 2.4, but that the economy is closed. Derive the optimal paths of consumption and capital accumulation in this case and provide an explanation of the difference between the Euler equation for this case and equation (7).

6. Foreign debt and trade surpluses.

(a) Using the relevant budget constraint, show that $b_0$, the initial value of external debt, is equal to the present value of net exports, provided an NPG condition is satisfied.

(b) Suppose that for some period of time a country's external debt is growing more rapidly than at the rate $r - n$. What can you conclude about the likelihood that the NPG condition will be violated in the long run? What then is the relevance of the NPG condition?

7. Suppose that in a closed economy there is an unexpected permanent reduction in the efficiency of production, represented in the symbols in the text as an increase in $z_0$. Assuming that the economy started in a steady state, derive and explain its optimal dynamic adjustment toward the new steady state.

8. Growth with increasing returns, I.

Consider an economy with the production function

$$Y = K^{a}N^{1-a}, \quad b > 0, a + b < 1$$

so that there are increasing returns to scale but decreasing returns to capital given labor. Population is growing at the rate $n$, and there is no depreciation.

(a) Show that it is possible for capital, output, and consumption all to grow at the same rate $g$. This is known as balanced growth. Derive the balanced growth rate $g$, and explain its dependence on $a$, $b$, and $n$.

(b) Suppose that the felicity function for the representative family is $u(c_t) = \ln c_t$ and that the family has a constant discount rate $\theta$.

Assuming that the economy converges to a balanced growth path, characterize the steady state marginal product of capital. Compare it to the modified golden rule level that would obtain under constant returns (i.e., with $b = 0$). Explain the difference.

9. Growth with increasing returns, II. (This follows Rebelo 1987.)

Consider the following economy: Population is constant and normalized to unity, and the representative individual maximizes

$$\int_0^\infty u(C) \exp(-\theta) \, dt.$$

$K$ is the capital stock in the economy and can be used either to produce consumption goods or new capital goods. Let $x, 0 \leq x \leq 1$, be the proportion of capital used in the production of consumption goods. The two production functions for consumption and investment goods are given by

$$C = F(xK),$$

$$F(0) = 0, \quad F'(\cdot) > 0, F''(\cdot) < 0.$$

$$dK/dt = l = B(1 - x)K; B$$ is a positive constant. Capital does not depreciate.

(a) What is the maximum growth rate of capital in this economy? What is the associated level of consumption?

(b) Derive the first-order conditions associated with this maximization problem. Interpret them. Give, in particular, an interpretation of the Lagrange multipliers and costate variables as shadow prices.

(c) Assume that $F(xK) = A(xK)^\alpha$, where $0 < \alpha < 1$, and that $U(C) = \ln(C)$. Show that if the economy converges to a balanced growth path, the rate of growth of consumption is given by $a(B - \theta)$. Explain in words.

What happens to the relative price of capital goods in terms of consumption goods along the balanced growth path?

(d) Contrast your results with those obtained in the conventional Ramsey model. Explain why they differ.

(e) How does this model do in terms of explaining the basic facts of growth as laid out by Kaldor and Solow, and summarized in chapter 1? What is the relation of consumption to income along the balanced growth path? What is the relation of output to capital? (Be careful about how you define capital—value or volume—here.)

Notes

1. In chapter 3 we show that people who have finite lives may still act as if they in effect had infinite lives.

2. Frank Ramsey was a Cambridge, England, mathematician and logician who died at the age of 26. His genius is evidenced by the fact that he had written three classic articles in economics by the age at which many economists are contemplating leaving graduate school. J. M. Keynes (1930) eulogizes Ramsey.

3. If depreciation is exponential at the rate $\lambda$, then gross output is $Y + \lambda K = F(K, N) + \lambda K = G(K, N)$. If $F(K, N)$ is degree one homogeneous, so is $G(K, N)$.

4. An alternative plausible formulation is the so-called Benthamite welfare function in which the felicity function becomes $N_t u(c_t)$ so that the number of family members receiving the given utility level is taken into account. Recognizing that $N_t = N_t e^\theta$, we see that the Benthamite formulation is equivalent to reducing the rate of time preference to $(\theta - n)$ because the larger size of the family at later dates in effect
increases the weight given to the utility of the representative individual in a later generation.

In assuming that \( \theta > 0 \), we depart from Ramsey who, interpreting the maximization problem as the problem solved by a central planner, argued that there was no ethical case for discounting the future.

5. Ordinary calculus optimization methods have to be augmented to handle the presence of a time derivative in constraint (2). Intriligator (1971) provides an introduction to intertemporal optimization methods.

6. A warning is in order here. First, under weaker assumptions than those made in the text, for example, a linear production function or no discounting, an optimum may not exist. Even if an optimum does exist, the transversality condition, equation (8), may not be necessary. But if one is ready to set sufficiently strong conditions for the maximization problem, these problems can usually safely be ignored. For a more careful statement and further discussion, see Shell (1969) and Benveniste and Scheinkman (1982).

7. Note from the formulation of the central planner's problem that it is implicitly assumed that capital can be consumed.

8. We emphasize again that as intuitive as this argument for the transversality condition is, there are infinite horizon problems in which the transversality condition is not necessary for the optimal path. See Shell (1969) and Michel (1982).

9. To show that the utility function converges to the logarithmic function as \( \gamma \) tends to unity, use L'Hospital's rule.

10. On the basic measures of risk aversion, see J. Pratt in Diamond and Rothschild (1978); see also the following articles in Diamond and Rothschild by Yaari and by Rothschild and Stiglitz.

Behavior toward risk and the degree of substitution between consumption at different times are conceptually two different issues. Under the assumption that the von Neumann-Morgenstern utility integral is additively separable over time, however, the two depend only on the curvature of the instantaneous utility function and are thus directly related. See chapter 6 for further discussion.

11. In steady state, with \( dk/dt = 0 \), we have from (2),

\[ c^* = f(k^*) - nk^*. \]

Maximization of \( c^* \) with respect to \( k^* \) gives the golden rule, that the marginal product of capital (or interest rate) is equal to the growth rate of population.

12. We freely interchange the marginal product and interest rates. We show later that in the decentralized Ramsey economy, the two are indeed equal.

13. The result that the steady state interest rate does not depend on the utility function can, however, be easily overturned. If labor-augmenting (Harrod-neutral) technical progress is taking place at the rate \( \mu \), so that

\[ Y_t = f(K, \exp(\mu t)N), \]

and if the utility function is of the CRRA class, then the modified golden rule condition becomes \( f'(k^*) = 0 + \sigma \mu + n. \) (In this case \( k^* \) is the ratio of capital to effective labor, i.e., \( K/\exp(\mu t)N \), and the steady state is one in which consumption per capita is growing at the rate \( \mu \).)

14. The analysis can also be undertaken in \((k, \lambda)\) space, using the first-order condition (6).

15. The behavior of consumption on the horizontal axis, where \( c = 0 \), depends on the value of the instantaneous elasticity of substitution \( \sigma(c) \) for \( c = 0 \). Equation (7) implies that

\[ \frac{dc}{dt} = \sigma(c)[f'(k) - \gamma - \eta]c. \]

If \( \sigma^{-1}(0) \) is not zero, then \( dc/dt = 0 \) when \( c = 0 \). We assume this to be the case. If the condition is not met, one must examine the behavior of \( c \sigma(c) \) at \( c = 0 \).

16. Throughout the book we will encounter phase diagrams in which there is only one convergent path. Although we will often simply assume that the economy proceeds on this converging path, an argument must be made in each case that the converging path is the only one that satisfies the conditions of the problem. As we will see in chapter 5, there are cases in which we cannot rule out some of the diverging paths.

17. Changes in \( f' \) and \( \theta \) affect both the rate of convergence to the steady state and the steady state capital stock itself.

18. The condition that the rental rate on capital is equal to the interest rate is special to this one-good model. If the relative price of capital, \( p_k \), could vary, asset market equilibrium would ensure that the expected rate of return from holding capital would be equal to the interest rate. The rate of return from holding capital is the rental rate, \( r \), plus any capital gains on capital minus depreciation, all expressed relative to the price of the capital:

\[ \text{rate of return} = \frac{r + (dp_k/dt) - \delta p_k}{p_k} = \text{real interest rate}, \]

where \( \delta \) is the rate of depreciation. In the single-good model, \( p_k \) is identically one, so there are no changes in the relative price of capital, and we are assuming that \( \delta \) is zero; accordingly, the rate of return on capital is \( r \), which is equal to the interest rate. (We are implicitly assuming that the economy never specializes completely; if it did not save at all, the relative price of capital goods could be less than one; if it did not consume at all, the relative price of capital could exceed one.)

19. For notational convenience we shall assume that there is just one family and one firm, both acting competitively.

20. There are many alternative ways of describing the decentralized economy. For example, firms can own the capital and finance investment by either borrowing or issuing equity. Or, instead of operating with spot factor markets, the economy may
operate in the Arrow-Debreu complete market framework in which markets for current and all future commodities, including services, are open at the beginning of time; all contracts are made then, and the rest of history merely executes these contracts. Under perfect foresight, all these economies will have the same allocation of resources.

21. We limit ourselves in what follows to paths of wages and rental rates such that the following condition is satisfied:

\[
\lim_{t \to \infty} \exp \left[ - \int_0^t (r_s - n) \, du \right] = 0.
\]

This condition says, roughly, that asymptotically the interest rate must exceed the rate of population growth. We will show that the equilibrium path indeed satisfies this condition. A complete argument would show that if this condition is not satisfied, there is no equilibrium. See note 25 below for further elaboration.

22. In the present model, in which all families are the same, they will in equilibrium have the same wealth position and hold the same fraction of the capital stock. Since the aggregate capital stock must be positive, each family will, in equilibrium, have positive wealth. This is, however, a characteristic of equilibrium, not a constraint that should be imposed a priori on the maximization problem of each family. In an economy with heterogeneous families, or families with different paths of labor income, positive aggregate capital may coexist with temporary borrowing by some families.

23. Charles Ponzi, one of Boston's sons, made a quick fortune in the 1920s using chain letters. He was sent to prison and died poor.

24. This raises the question of how the no-Ponzi-game condition is actually enforced. The fact that parents cannot, for the most part, leave negative bequests to their children implies that family debt cannot increase exponentially. It may in fact impose a stronger restriction on borrowing than the no-Ponzi-game condition used here.

25. Following up on note 21, there is one loose end in our proof of equivalence, which we now tie up. We have restricted ourselves to paths where the interest rate exceeds asymptotically the population growth rate. Given this restriction, we showed that there is an equilibrium path, which is the same as the central planning one, so that \( r \) converges asymptotically to \( n + \theta \). We now need to show that paths on which the interest rate is asymptotically less than \( n \) cannot be equilibria. To see why, rewrite the budget constraint facing the family as

\[
\frac{da_t}{dt} = (r_t - n)a_t + (c_t - w_t).
\]

Consider then two paths of consumption, which have the same level of consumption after some time \( T \), so that \( c_T - w_T \) is the same on both paths after \( T \). Then, if \( r_T - n \) is asymptotically negative, both paths will lead to the same asymptotic value of \( a \) (the same level of net indebtedness if \( a \) is negative). If one path satisfies the no-Ponzi-game condition, so will the other. But this implies that the family will always want to have very high (possibly infinite) consumption until time \( T \). This cannot be an equilibrium.

26. We consider endogenous government spending in chapter 11.

27. Government spending, for instance, on education, might substitute for private spending, in which case the utility function would have to be amended appropriately. Similarly, government spending on defense and public safety might contribute to the economy's productive capacity, but we do not model any such effects.

28. The dynamics of investment and savings in a closed economy with adjustment costs are studied in Abel and Blanchard (1983).

29. Blanchard (1983), Fischer and Frenkel (1972), and Svensson (1984) have used similar models to examine the dynamics of foreign debt and the current account.

30. Investment decisions based on adjustment costs have been modeled by Abel (1981), Eisner and Strotz (1963), Lucas (1967), and Tobin (1969). Our specification is that of Hayashi (1982).

31. The conditions specified after equation (31) ensure the properties of the installation cost function \( I(T/k) \). Note that, in practice, when capital depreciates, the costs of small rates of disinvestment, which can take place through depreciation, are likely to be very small or zero.

Instead of defining both a production and an installation cost function, we could have defined a 'net' production function that gives output available for consumption or export, \( H(K, N, I) \). This is the approach taken, for example, by Lucas (1967). In our case \( H(K, N, I) = F(K, N) - I[1 + T(UI)] \), where uppercase letters are total amounts of corresponding per capita variables. The function \( H(\cdot) \) has constant returns to scale if \( F(\cdot) \) does.

32. If the world interest rate had differed from the rate of time preference, the country would either accumulate or decumulate forever. This follows from the Euler equation in the absence of population growth, which from section 2.2 will give \( [du(c_t)/dt/u'(c_t) = 0 - r \), where \( r \) is the interest rate. If the country accumulates forever because \( \theta < r \), then it eventually becomes a large economy and begins to affect the world interest rate; if \( \theta > r \), then the country runs its wealth down as fast as it can. To avoid these difficulties, we set \( \theta = r \). We could also obtain convergence to a steady state if we specified a time path for the world interest rate that converges to \( \theta \), rather than always being equal to \( \theta \). We assume \( r = \theta \) for simplicity.

33. We state the NPG condition as an equality. We could again state it as an inequality, requiring the present discounted value of debt to be nonnegative. But if marginal utility is positive, the central planner will not want to accumulate increasing claims on the rest of the world forever. Thus the NPG condition will hold with equality.
34. Defining the costate variable on (31) as \( \mu(t) \exp(-\theta t) \) rather than as a single variable is a matter of convenience, as will become clear later when we show that \( q \) plays a key role in determining investment.

35. Note that because of the equality of the interest rate and the subjective discount rate, the marginal propensity to consume out of wealth is equal to \( \theta \) independently of the form of the felicity function.

36. Note that given constant \( \mu \), equation (39) implies that \( \lim n(t) \exp(-\theta t) = 0 \) as \( t \) goes to \( \infty \). This is, however, not the same as \( \lim q, \exp(-\theta t) = 0 \) as \( t \) goes to \( \infty \), which is the condition needed to derive (44). To derive (44), one must characterize the phase diagram associated with equations (37') and (43) and show that the only path that satisfies these equations and the transversality condition (39) is a path where both \( k \) and \( q \) tend to \( k^* \) and \( q^* \), respectively, so that \( \lim q, \exp(-\theta t) = 0 \) as \( t \) goes to \( \infty \).

37. This way of thinking about the investment decision was developed by Tobin. For that reason, \( q \) is often called Tobin's \( q \). See Hayashi (1982) for a discussion of the relation of \( q \) to its empirical counterparts; in particular, Hayashi discusses the conditions under which average \( q \), as reflected, say, in the stock market valuation of a firm, is equal to marginal \( q \), the shadow value of an additional unit of installed capital. Marginal and average \( q \) are equal, leaving aside tax issues, if the firm's production function and the adjustment cost function \( T(t) \) are each first-degree homogeneous and firms operate in competitive markets. Under those assumptions one would expect a tight relation between the market valuation of firms and their investment decisions. Empirically, although average \( q \) and investment rates are indeed correlated, the relation is far from tight (see Hayashi 1982).

38. If there is population growth at the rate \( n \), then \( q^* \) is given by \( n = \phi(q^*) \) so that \( q^* = f(k^*) - n^2 T(n) \).

39. The restriction to local dynamics ensures that \( dq/dt = 0 \) is negatively sloped; away from the steady state there is no assurance that the slope of \( dq/dt = 0 \) is negative without imposing more conditions on the \( T(t) \) function. However, the restrictions imposed on \( T(t) \) are sufficient to ensure that there is a unique steady state in the neighborhood of which the \( dq/dt = 0 \) locus is negatively sloped.

40. In appendix D we show that the transversality condition suffices in the linearized system to rule out any divergent paths that satisfy the necessary conditions (47).

41. The current account always has present discounted value equal to zero when condition (32) is satisfied; it is only when the initial debt is zero that the same applies to the trade account.

42. Given the equivalence between the command optimum and the decentralized economy, the shocks can also be interpreted as taxes, where the government is using the proceeds of the taxes to finance government spending that does not affect the utility function, as in section 2.3.

43. This experiment raises the methodological issue of how unexpected changes can occur in a model in which there is perfect foresight. The correct way to analyze such changes would be to set up the maximizing problems of the central planner or economic agents explicitly as decision problems under uncertainty. This substantially complicates the analysis, and we defer this to chapter 6; we can think of the approach taken here as a shortcut in which the surprise is an event that was regarded as so unlikely as not to be taken into account up to the time it occurs.

44. If individuals dislike changes in the rate of consumption so that the felicity function is, for instance, \( \delta(c, dc/dt) \), the reduction in \( z_1 \) would cause a smaller decline in consumption than in output initially; the country would in that case initially borrow abroad temporarily to cushion the shock of the reduction in the standard of living, and end up with permanently higher debt and lower consumption.

45. We briefly return in chapter 7 to the issue of nonseparability in the context of a discussion of labor supply.

As we shall see, however, the distinction between the felicity function and the discount factor becomes somewhat blurred when we allow for more general formulations of this discount factor.

46. The argument to this point does not eliminate the possibility that there is no steady state. The argument of this paragraph can be seen, however, to imply the existence of a steady state with \( r = \theta^\mu \).

47. The no-Ponzi-game condition prevents the shortsighted from going further and further into debt.

48. Ramsey (1928) conjectured this result; it was proved by Becker (1980).

49. Note the similarity between the discussion here and that of the relationship between the world interest rate and rate of time preference of a small country in section 2.4.

50. Because the point we are about to make about the optimal program does not depend on the presence of \( x(t) \) in the discount function, we omit that argument henceforth.

51. This result is due to Strotz (1956).

52. An example is \( D(\cdot) = \max[0, A - \theta(t - s)] \).

53. See Elster (1979) and Schelling (1984) for more extensive discussion of how people do and should deal with inconsistencies. Issues of time consistency also arise in the context of games between agents or between agents and the government. We will study these in chapter 11.

54. There is no "correct" way to behave when tastes are dynamically inconsistent, for there is no way of knowing which is the right set of tastes: the title "Ulysses and the Sirens" (Elster 1979) refers to Ulysses's strategy of having himself tied to the mast to avoid succumbing to the Sirens' cry—but maybe the real Ulysses was the one who would have succumbed if the other Ulysses hadn't tied him to the mast.
55. Epstein and Hynes (1983) suggest an alternative specification, namely,
\[ \int_0^t \exp \left[ - \int_0^t u'(c_t) \, dc_t \right] \, ds. \]
This specification has the same qualitative implication as Uzawa’s but is more tractable analytically. Note that in this form there is no longer any distinction between the discount rate and the instantaneous felicity function.

56. Lucas and Stokey (1984) work with a model of this type.

57. Once again, there are many alternative ways of describing the decentralized economy. Firms could, instead, finance investment by issuing shares or by borrowing either abroad or domestically. The real allocation would be the same in all cases.

58. If investment is so high that net cash flows are negative, the firm is, in effect, issuing equity by paying a negative dividend, that is, making a call on stockholders for cash.

References


Chapter 2

The Overlapping Generations Model

The overlapping generations model of Allais (1947), Samuelson (1958), and Diamond (1965) is the second basic model used in micro-based macroeconomics. The name implies the structure: at any one time individuals of different generations are alive and may be trading with one another, each generation trades with different generations in different periods of its life, and there are generations yet unborn whose preferences may not be registered in current market transactions.

The model is widely used because it makes it possible to study the aggregate implications of life-cycle saving by individuals. The capital stock is generated by individuals who save during their working lives to finance their consumption during retirement. The determinants of the aggregate capital stock as well as the effects of government policy on the capital stock and the welfare of different generations are easily studied. The model can be extended to allow for bequests, both intentional and unintentional.

Given the descriptive appeal of the life-cycle hypothesis, these uses of the model would alone justify its widespread popularity. But beyond that, the model provides an example of an economy in which the competitive equilibrium is not necessarily that which would be chosen by a central planner. There is an even stronger result: the competitive equilibrium may not be Pareto optimal. Life-cycle savers may overaccumulate capital, leading to equilibria in which everyone can be made better off by consuming part of the capital stock. This possible inefficiency of the equilibrium contrasts sharply with the intertemporal efficiency of the competitive equilibrium in the Ramsey model. One of the goals of this chapter is to elucidate the aspects of the life-cycle model that make inefficiency possible.

We start, in section 3.1, with the simplest version of the overlapping generations model in which individuals live for only two periods. Starting from individual maximization, we show how the aggregate capital stock evolves over time. We extend the model to consider the effects of altruism...